

\*\*[JAC(2010-06-27)]\*\*

# ON UNRAMIFIED EXTENSIONS OF A POLYNOMIAL RING OVER A FIELD OF CHARACTERISTIC ZERO

SUSUMU ODA

ABSTRACT. We show the Jacobian Conjecture : If  $f_1, \dots, f_n$  be elements in a polynomial ring  $k[X_1, \dots, X_n]$  over a field  $k$  of characteristic zero such that  $\det(\partial f_i / \partial X_j)$  is a nonzero constant, then  $k[f_1, \dots, f_n] = k[X_1, \dots, X_n]$ .

## 1. INTRODUCTION

Let  $k$  be an algebraically closed field, let  $\mathbb{A}_k^n = \text{Max}(k[X_1, \dots, X_n])$  be an affine space of dimension  $n$  over  $k$  and let  $f : \mathbb{A}_k^n \rightarrow \mathbb{A}_k^n$  be a morphism of affine spaces over  $k$  of dimension  $n$ . Then  $f$  is given by

$$\mathbb{A}_k^n \ni (x_1, \dots, x_n) \mapsto (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)) \in \mathbb{A}_k^n,$$

where  $f_i(X_1, \dots, X_n) \in k[X_1, \dots, X_n]$ .

If  $f$  has an inverse morphism, then the Jacobian  $\det(\partial f_i / \partial X_j)$  is a nonzero constant. This follows from the easy chain rule. The Jacobian Conjecture asserts the converse.

### Examples.

- (i) If  $k$  is of characteristic  $p > 0$  and  $f(X) = X + X^p$ , then  $df/dX = f'(X) = 1$  but  $X$  can not be expressed as a polynomial in  $f$ . Thus we must assume the characteristic of  $k$  is zero.
- (ii) Let  $k$  be a field of characteristic  $p > 0$  and let  $i : k[X, Y + XY^p] \hookrightarrow k[X, Y]$  be polynomial rings. Put  $F_1 = X, F_2 = Y + XY^p$ . Then  ${}^ai : \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^2$  is unramified because the Jacobian of  ${}^ai$  is invertible. We see that  $k[X, Y] = k[F_1, F_2][Y]$  and

---

2000 *Mathematics Subject Classification* : Primary 13B25, Secondary 13M10

*Key words and phrases*: unramified, etale, CM-rings, the Picard groups, the Jacobian Conjecture and polynomial rings

the equation of minimal degree of  $Y$  over  $k[F_1, F_2]$  is  $F_1 Y^p + Y - F_2 = 0$ , so that  $Y$  is not integral over  $k[F_1, F_2]$  ([6]).

The algebraic form of the Jacobian Conjecture is the following :

**The Jacobian Conjecture in algebraic form.** *If  $f_1, \dots, f_n$  be elements in a polynomial ring  $k[X_1, \dots, X_n]$  over a field  $k$  of characteristic zero such that  $\det(\partial f_i / \partial X_j)$  is a nonzero constant, then  $k[f_1, \dots, f_n] = k[X_1, \dots, X_n]$ .*

We can also say it in terms of algebraic geometry as follows:

**The Jacobian Conjecture in geometric form.** *Let  $f : \mathbb{A}_k^n \rightarrow \mathbb{A}_k^n$  be a morphism of affine spaces of dimension  $n$  ( $n \geq 1$ ) over a field  $k$  of characteristic zero. If  $f$  is unramified, then  $f$  is an isomorphism.*

Throughout this paper, all fields, rings and algebras are assumed to be commutative with unity. For a ring  $R$ ,  $R^\times$  denotes the set of units of  $R$  and  $K(R)$  the total quotient ring.  $\text{Spec}(R)$  denotes the affine scheme defined by  $R$  or merely the set of all prime ideals of  $R$  and  $\text{Ht}_1(R)$  denotes the set of all prime ideals of height one. Our general reference for unexplained technical terms is [12].

Our objective of this paper is to prove the Jacobian Conjecture.

## 2. PRELIMINARIES

**Definition (Unramified)** Let  $f : A \rightarrow B$  be a ring-homomorphism of finite type of Noetherian rings. The homomorphism  $f$  is called *unramified* at  $P \in \text{Spec}(B)$  if  $PB_P = (P \cap A)B_P$  and  $k(P) = B_P/PB_P$  is a finite separable field extension of  $k(P \cap A) = A_{P \cap A}/(P \cap A)A_{P \cap A}$ . The ring  $B$  is called *unramified* over  $A$  if  $f_P : A_{P \cap A} \rightarrow B_P$  is unramified for all  $P$  of  $B$ . The homomorphism  $f_P : A_{P \cap A} \rightarrow B_P$  is called *etale* at  $P$  if  $f_P$  is unramified and flat, and  $f$  is *etale* over  $A$  if  $f_P$  is etale for all  $P \in \text{Spec}(B)$ . The morphism  ${}^a f : \text{Spec}(B) \rightarrow \text{Spec}(A)$  is called *unramified* (resp. *etale*) if  $f : A \rightarrow B$  is unramified (resp. etale).

**Definition (Simply Connected)** An integral domain  $R$  is called *simply connected* if the following condition holds:

Provided any domain  $A$  with an etale (module-)finite ring-homomorphism  $\varphi : R \rightarrow A$ ,  $\varphi$  is an isomorphism.

The following assertion is called the *simply connectivity* of affine space  $\mathbb{A}_k^n$  ( $n \in \mathbb{N}$ ) over field  $k$  of characteristic zero. Its algebraic proof is seen in [19].

**Proposition 2.1.** ([19]) *Let  $k$  be an algebraically closed field of characteristic zero and let  $B$  be a polynomial ring  $k[Y_1, \dots, Y_n]$ . Let  $D$  be a  $k$ -affine domain. If  $D$  is finite etale over  $B$  then  $D = B$ .*

Recall the following well-known results, for convenience sake.

**Lemma 2.2** ([12, (21.D)]). *Let  $(A, m, k)$  and  $(B, n, k')$  be Noetherian local rings and  $\phi : A \rightarrow B$  a local homomorphism (i.e.,  $\phi(m) \subseteq n$ ). If  $\dim B = \dim A + \dim B \otimes_A k$  holds and if  $A$  and  $B \otimes_A k = B/mB$  are regular, then  $B$  is flat over  $A$  and regular.*

*Proof.* If  $\{x_1, \dots, x_r\}$  is a regular system of parameters of  $A$  and if  $y_1, \dots, y_s \in n$  are such that their images form a regular system of parameters of  $B/mB$ , then  $\{\varphi(x_1), \dots, \varphi(x_r), y_1, \dots, y_s\}$  generates  $n$ . and  $r + s = \dim B$ . Hence  $B$  is regular. To show flatness, we have only to prove  $\text{Tor}_1^A(k, B) = 0$ . The Koszul complex  $K_*(x_1, \dots, x_r; A)$  is a free resolution of the  $A$ -module  $k$ . So we have  $\text{Tor}_1^A(k, B) = H_1(K_*(x_1, \dots, x_r; A) \otimes_A B) = H_1(K_*(x_1, \dots, x_r; B))$ . Since the sequence  $\varphi(x_1), \dots, \varphi(x_r)$  is a part of a regular system of parameters of  $B$ , it is a  $B$ -regular sequence. Thus  $H_i(K_*(x_1, \dots, x_r; B)) = 0$  for all  $i > 0$ .  $\square$

From Lemma 2.2, every unramified homomorphisms are etale in the case affine regular domains over fields. We will use the following corollary of Lemma 2.2 in the section 4.

**Corollary 2.3.** *Let  $k$  be a field and let  $R$  be a  $k$ -affine regular domain. Let  $S$  be a finitely generated ring-extension domain of  $R$ . If  $S$  is unramified over  $R$ , then  $S$  is etale over  $R$ , that is, for any  $P \in \text{Spec}(S)$ ,  $S_P$  is etale over  $R_{P \cap R}$ .*

*Proof.* We have only to show that  $S$  is flat over  $R$ . Note first that  $K(S)$  is finite algebraic over  $K(R)$  because the domain  $S$  is unramified over the domain  $R$ . Take any maximal ideal  $M \in \text{Spec}(S)$  and put  $m = M \cap R$ . Then  $R_m \hookrightarrow S_M$  is a local homomorphism. Since  $S_M$  is unramified over  $R_m$ , we have  $\dim S_M = \dim R_m$  because both  $R$  and  $S$  are  $k$ -affine domain of the same dimension. Since  $S_M$  is unramified over  $R_m$ , we have  $S_M \otimes_{R_m} k(m) = S_M / (mR_m)S_M = S_M / MS_M$  is a field. So by Lemma 2.2,  $S_M$  is flat over  $R_m$ . Therefore  $S$  is flat over  $R$  by [4].  $\square$

Let  $J$  be a non-zero ideal of an integral domain  $A$  and let  $J^{-1} := \{\alpha \in K(A) \mid \alpha J \subseteq A\}$ , which is a fractional ideal in  $K(A)$ . It is easy to see that  $A \subseteq J^{-1} \subseteq K(A)$ .

Let  $R$  be an integral domain with quotient field  $K(R)$  and let  $J$  be an ideal of  $R$ . The set  $\text{Trans}(J; R) := \{\alpha \in K(R) \mid \alpha J^n \subseteq R\}$  is a subring of  $K(R)$  containing  $R$ , which called  $J$ -Transform of  $J$ . Let  $J^{-n} = \{\alpha \in K(R) \mid \alpha J^n \subseteq R\}$ . Then  $\text{Trans}(J; R) = \bigcup_{n \in \mathbb{N}} J^{-n}$ . For any  $n \in \mathbb{N}$ , it holds that  $(J^n)^{-1} = J^{-n}$  by definition.

**Lemma 2.4** ([14, p.51, Theorem 3']). *Let  $k$  be a field and let  $V$  be a  $k$ -affine variety defined by a  $k$ -affine ring  $R$  (which means a finitely generated algebra over  $k$ ) and let  $F$  be a closed subset of  $V$  defined by a non-zero ideal  $J$  ( $\neq R$ ) of  $R$ . Then the variety  $V \setminus F$  is  $k$ -affine if and only if  $1 \in J\text{Trans}(J; R)$ . In this case,  $F$  is pure of codimension one,  $\text{Trans}(J; R)$  is the affine ring of  $V \setminus F$ , and  $\text{Trans}(J; R) = R[J^{-n}]$  for some  $n \in \mathbb{N}$ .*

The following Zariski's Theorem (ZMT for short) will be used in the next section. ZMT is often expressed in other statements.

**Lemma 2.5** ([13])(Zariski's Main Theorem). *Let  $A$  be an integral domain and let  $B$  be an  $A$ -algebra of finite type which is quasi-finite over  $A$ . Let  $\overline{A}$  be the integral closure of  $A$  in  $B$ . Then the canonical morphism  $\text{Spec}(B) \rightarrow \text{Spec}(\overline{A})$  is an open immersion.*

**Lemma 2.6** ([13]). *Let  $k$  be an algebraically closed field, let  $R$  be a  $k$ -affine domain and let  $L$  be a finite algebraic field extension of  $K(R)$ . Then the integral closure  $R_L$  of  $R$  in  $L$  is a finitely generated  $R$ -module.*

### 3. SOME KEY-STEPS TO THE GOAL

We make an ordered set  $\mathbb{Z} \cup \{\infty\}$  by adding to  $\mathbb{Z}$  an element  $\infty$  bigger than all the elements of  $\mathbb{Z}$ , and fix the convention  $\infty + n = \infty$  for  $n \in \mathbb{Z}$  and  $\infty + \infty = \infty$ . A map  $v : K \rightarrow \mathbb{Z} \cup \{\infty\}$  from a field  $K$  to  $\mathbb{Z} \cup \{\infty\}$  is called a *discrete (additive) valuation* of  $K$  if it satisfies the conditions:

- (1)  $v(xy) = v(x) + v(y)$  ;
- (2)  $v(x + y) \geq \min\{v(x), v(y)\}$  ;
- (3)  $v(0) = \infty$  if and only if  $x = 0$ .

We set

$$R_v = \{x \in K \mid v(x) \geq 0\}, \text{ and } m_v = \{x \in K \mid v(x) > 0\}$$

obtaining a valuation ring  $R_v$  of  $K$  with  $m_v$  as its maximal ideal, and call  $R_v$  the discrete valuation ring (the DVR for short) of  $v$ , and  $m_v$  the valuation ideal of  $v$ . Note that the ring  $R_v$  is a local ring and  $m_v$  is the maximal ideal of  $R_v$  which is a principal ideal of  $R_v$  and that in (2), if  $v(x) < v(y)$  then  $v(x + y) = v(x)$ . If, however,  $v(x) = v(y)$ , we can choose some  $a \in k$  such that  $v(x + ay) = v(x) = v(ay)$  when  $R$  contains an infinite field  $k$ . We say a ring  $R$  isomorphic to a DVR  $R_v$  for a discrete valuation  $v$  as above is also called a DVR.

**Lemma 3.1.** *Let  $C$  be a Noetherian normal ring. Then  $C_p$  is a DVR for every  $p \in \text{Ht}_1(C)$ . (Write  $v_p$  for its associated valuation.)*

**Lemma 3.2.** *Let  $C \hookrightarrow T$  be Noetherian domains. Assume that*

- (1)  *$C$  is a normal domain and  $T$  is a UFD;*
- (2)  *$\text{Spec}(T) \rightarrow \text{Spec}(C)$  is an open immersion;*
- (3) *the closed subset  $\text{Spec}(C) \setminus \text{Spec}(T) = V(I)$  with  $I = \sqrt{I}$  an ideal of  $C$ , is pure*

of codimension one.

Then there exists an element  $x \in I$  such that  $x \in T^\times$ .

*Proof.* Note that any  $w \in C$ ,  $v_p(w) \geq 0$  for all  $p \in \text{Ht}_1(C)$  and that any prime divisor of a principal ideal of  $C$  is of height one because  $C$  is normal (and so  $C$  satisfies Serre's conditions  $(R_1)$  and  $(S_2)$ ). Take  $y \in I$  ( $y \neq 0$ ). Put  $\Delta = V(I) \cap \text{Ht}_1(C)$  and  $\overline{\Delta} = \text{Ht}_1(C) \setminus \Delta$ . Since  $y \in I$ , any  $p \in \Delta$  contains  $y$ , that is,  $v_p(y) \geq 1$  for all  $p \in \Delta$ . We know that every  $P \in \text{Ht}_1(T)$ ,  $T_P = C_{P \cap C}$  with  $P \cap C \in \overline{\Delta}$  and for  $p \in \overline{\Delta}$ , there exists only one  $P \in \text{Ht}_1(T)$  such that  $P \cap C = p$  because  $\text{Spec}(T) \rightarrow \text{Spec}(C)$  is an open immersion. So the associated valuations  $v_P(\ )$  and  $v_p(\ )$  are considered the same valuation on  $K(T) = K(C)$  for  $p \in \overline{\Delta}$ .

Let  $U = \{q_k | q_k \in \overline{\Delta}, v_{q_k}(y) = n_k > 0 (1 \leq k \leq m)\}$ , which is a finite set. Then  $P_k \cap C = q_k$  for  $P_k \in \text{Ht}_1(T)$  ( $1 \leq k \leq m$ ). Note that  $P_k = t_k T$  ( $t_k \in T$ ), a principal ideal of  $T$  because  $T$  is a UFD. Since  $t_k \notin pC_p$  for  $p \in \Delta$ , it follows that  $1/t_k$  is contained in  $C_p$  and  $v_p(t_k) = 0$  for  $p \in \Delta$ .

Put  $x = y / (\prod_{k=1}^m t_k^{n_k})$ . Then  $v_{P_k}(x) = v_{q_k}(x) = 0$  for all  $q_k = P_k \cap C \in U$  and hence  $v_q(x) = 0$  for all  $q \in \overline{\Delta}$ . Furthermore  $v_p(x) = v_p(y) \geq 1$  for all  $p \in \Delta$ , which also means that  $x \in I \subseteq C$ . Since  $v_{Q \cap C}(x) = v_Q(x) = 0$  for  $Q \in \text{Ht}_1(T)$  by the construction of  $x$ , we have  $xT = T$ , which means that  $x \in T^\times$ .  $\square$

**Theorem 3.3.** *Let  $S \hookrightarrow C \hookrightarrow T$  be extensions of integral domains over an algebraically closed field  $k$ . Assume that*

- (1)  $K(T)$  is finite algebraic over  $K(S)$ ;
- (2)  $S = k[Y_1, \dots, Y_n]$  is a polynomial ring over  $k$  and  $T$  is a  $k$ -affine UFD;
- (3)  $C$  is the integral closure of  $S$  in  $K(T)$ ;
- (4)  $\text{Spec}(T) \rightarrow \text{Spec}(C)$  is an open immersion;
- (5)  $T^\times = k^\times$ .

Then  $T = C$ .

*Proof.* Since  $T$  is normal, we have  $C \subseteq T$  and trivially  $K(C) = K(T)$ . Note that  $C$  is (module-)finite over  $S$  by Lemma 2.6. Let  $V(I) = \text{Spec}(C) \setminus \text{Spec}(T)$  for a radical ideal  $I$  of  $C$ . Then any prime divisor of  $I$  belongs to  $\text{Ht}_1(C)$  [13, p.51, Theorem 3']. Then there exists an element  $x \in I$  such that  $x \in T^\times$  by Lemma 3.2, which means  $x \in T^\times = k^\times$ . Therefore  $I = C$  and hence  $T = C$ .  $\square$

## 4. THE JACOBIAN CONJECTURE

The Jacobian Conjecture mentioned in the introduction of this paper has been settled affirmatively in several cases. Let  $k$  denote a field of characteristic zero. For example,

**Case(1)**  $k(X_1, \dots, X_n) = k(f_1, \dots, f_n)$  (cf. [6]);

**Case(2)**  $k(X_1, \dots, X_n)$  is a Galois extension of  $k(f_1, \dots, f_n)$  (cf. [6], [7] and [19]);

**Case(3)**  $\deg f_i \leq 2$  for all  $i$  (cf. [17] and [18]);

**Case(4)**  $k[X_1, \dots, X_n]$  is integral over  $k[f_1, \dots, f_n]$  (cf. [6]).

A general reference for the Jacobian Conjecture is [6].

Now we come to our main result.

**Remark 4.1.** (i) *To show the Jacobian Conjecture, we have only to show that the inclusion  $k[f_1, \dots, f_n] \longrightarrow k[X_1, \dots, X_n]$  is surjective. For this it suffices that  $k'[f_1, \dots, f_n] \longrightarrow k'[X_1, \dots, X_n]$  is surjective, where  $k'$  denotes an algebraic closure of  $k$ . Indeed, once we proved  $k'[f_1, \dots, f_n] = k'[X_1, \dots, X_n]$ , we can write for each  $i = 1, \dots, n$ :*

$$X_i = F_i(f_1, \dots, f_n),$$

where  $F_i(Y_1, \dots, Y_n) \in k'[Y_1, \dots, Y_n]$ , a polynomial ring in  $Y_i$ . Let  $L$  be an intermediate field between  $k$  and  $k'$  which contains all the coefficients of  $F_i$  and is a finite Galois extension of  $k$ . Let  $G = G(L/k)$  be its Galois group and put  $m = \#G$ . Then  $G$  acts on a polynomial ring  $L[X_1, \dots, X_n]$  such that  $X_i^g = X_i$  for all  $i$  and all  $g \in G$  that is,  $G$  acts on coefficients of an element in  $L[X_1, \dots, X_n]$ . Hence

$$mX_i = \sum_{g \in G} X_i^g = \sum_{g \in G} F_i^g(f_1^g, \dots, f_n^g) = \sum_{g \in G} F_i^g(f_1, \dots, f_n).$$

Since  $\sum_{g \in G} F_i^g(Y_1, \dots, Y_n) \in k[Y_1, \dots, Y_n]$ , it follows that  $\sum_{g \in G} F_i^g(f_1, \dots, f_n) \in k[f_1, \dots, f_n]$ . Therefore  $X_i \in k[f_1, \dots, f_n]$  because  $L$  has a characteristic zero. So we may assume that  $k$  is algebraically closed.

(ii) Let  $k$  be a field, let  $k[X_1, \dots, X_n]$  denote a polynomial ring and let  $f_1, \dots, f_n \in k[X_1, \dots, X_n]$ . If the Jacobian  $\det \left( \frac{\partial f_i}{\partial X_i} \right) \in k^\times (= k \setminus (0))$ , then  $k[X_1, \dots, X_n]$  is

unramified over the subring  $k[f_1, \dots, f_n]$ . Consequently  $f_1, \dots, f_n$  is algebraically independent over  $k$ . In fact, put  $T = k[X_1, \dots, X_n]$  and  $S = k[f_1, \dots, f_n] (\subseteq T)$ . We have an exact sequence by [12, (26.H)]:

$$\Omega_{S/k} \otimes_S T \xrightarrow{v} \Omega_{T/k} \rightarrow \Omega_{T/S} \rightarrow 0,$$

where

$$v(df_i \otimes 1) = \sum_{j=1}^n \frac{\partial f_i}{\partial X_j} dX_j \quad (1 \leq i \leq n).$$

So  $\det \left( \frac{\partial f_i}{\partial X_j} \right) \in k^\times$  implies that  $v$  is an isomorphism. Thus  $\Omega_{T/S} = 0$  and hence  $T$  is unramified over  $S$  by [4, VI, (3.3)] or [12]. Moreover  $K(T)$  is algebraic over  $K(S)$ , which means that  $f_1, \dots, f_n$  are algebraically independent over  $k$ .

Since a polynomial ring over a field of characteristic zero is simply connected by Proposition 1.2 we can prove the original Jacobian Conjecture as follows. We may assume a field  $k$  below may be assumed an algebraically closed by Remark 4.1(i).

**Theorem 4.2** (The Jacobian Conjecture). *Let  $f_1, \dots, f_n$  be elements in a polynomial ring  $k[X_1, \dots, X_n]$  over a field  $k$  of characteristic zero such that the Jacobian  $\det(\partial f_i / \partial X_j)$  is a nonzero constant. Then  $k[f_1, \dots, f_n] = k[X_1, \dots, X_n]$ .*

*Proof.* Put  $S = k[f_1, \dots, f_n]$  and  $T = k[X_1, \dots, X_n]$ . The Jacobian condition implies unramifiedness by Remark 4.1(ii). Note that  $\text{Spec}(T) \rightarrow \text{Spec}(S)$  is étale by Corollary 2.3 (cf. [6, p.296]). Let  $C$  denote the integral closure of  $S$  in  $K(T)$ . Then  $C$  is a finitely generated  $S$ -module by Lemma 2.6 and hence a  $k$ -affine domain, and  $C \subseteq T$  because  $T$  is a regular domain. It is trivially seen that  $K(C) = K(T)$ . By Zariski's Main Theorem (Lemma 2.5), we have  $\text{Spec}(T) \hookrightarrow \text{Spec}(C) \rightarrow \text{Spec}(T)$  which is induced from the injections  $S \hookrightarrow C \hookrightarrow T$ , where  $\text{Spec}(T) \rightarrow \text{Spec}(C)$  is an open immersion and  $\text{Spec}(C) \rightarrow \text{Spec}(S)$  is finite. From Theorem 3.3, we have  $T = C$ . Therefore  $T = C$  is finite and étale over  $S$ . Since the polynomial ring  $k[f_1, \dots, f_n]$  is simply connected, we conclude  $T = C = S$  by Proposition 2.1. **[END]**  $\square$



## 5. GENERALIZATION OF THE JACOBIAN CONJECTURE

The Jacobian Conjecture for  $n$ -variables can be generalized as follows.

**Theorem 5.1.** *Let  $A$  be an integral domain whose quotient field  $K(A)$  is of characteristic zero. Let  $f_1, \dots, f_n$  be elements of a polynomial ring  $A[X_1, \dots, X_n]$  such that  $\det(\partial f_i / \partial X_j)$  is a nonzero constant. Then  $A[X_1, \dots, X_n] = A[f_1, \dots, f_n]$ .*

*Proof.* We know that  $K(A)[X_1, \dots, X_n] = K(A)[f_1, \dots, f_n]$  by Theorem 4.2. It suffices to prove  $X_1, \dots, X_n \in A[f_1, \dots, f_n]$ . Hence

$$X_1 = \sum c_{i_1 \dots i_n} f_1^{i_1} \cdots f_n^{i_n}$$

with  $c_{i_1 \dots i_n} \in K(A)$ . If we set  $f_i = a_{i1}X_1 + \dots + a_{in}X_n +$  (higher degree terms),  $a_{ij} \in A$ . then the assumption implies that the determinant of a matrix  $(a_{ij})$  is a unit in  $A$ . Let

$$Y_i = a_{i1}X_1 + \dots + a_{in}X_n \quad (1 \leq i \leq n).$$

Then  $A[X_1, \dots, X_n] = A[Y_1, \dots, Y_n]$  and  $f_i = Y_i +$  (higher degree terms). So to prove the assertion, we can assume that without loss of generality the linear parts of  $f_1, \dots, f_n$  are  $X_1, \dots, X_n$ , respectively. Now we introduce a linear order in the set  $\{(i_1, \dots, i_n) \mid i_k \in \mathbf{Z}\}$  of lattice points in  $\mathbf{R}^n$  (where  $\mathbf{R}$  denotes the field of real numbers) in the way :  $(i_1, \dots, i_n) > (j_1, \dots, j_n)$  if (1)  $i_1 + \dots + i_n > j_1 + \dots + j_n$  or (2)  $i_1 + \dots + i_k > j_1 + \dots + j_k$  and  $i_1 + \dots + i_{k+1} = j_1 + \dots + j_{k+1}, \dots, i_1 + \dots + i_n = j_1 + \dots + j_n$ . We shall show that every  $c_{i_1 \dots i_n}$  is in  $A$  by induction on the linear order just defined. Assume that every  $c_{j_1 \dots j_n}$  with  $(j_1, \dots, j_n) < (i_1, \dots, i_n)$  is in  $A$ . Then the coefficients of the polynomial in  $f_1, \dots, f_n$

$$\sum c_{j_1 \dots j_n} f_1^{j_1} \cdots f_n^{j_n}$$

are in  $A$ , where the summation ranges over  $(j_1, \dots, j_n) \geq (i_1, \dots, i_n)$ . In this polynomial, the term  $X_1^{i_1} \cdots X_n^{i_n}$  appears once with the coefficient  $c_{i_1 \dots i_n}$ . Hence  $c_{i_1 \dots i_n}$  must be an element of  $A$ . So  $X_1$  is in  $A[f_1, \dots, f_n]$ . Similarly  $X_2, \dots, X_n$  are in  $A[f_1, \dots, f_n]$  and the assertion is proved completely.  $\square$

**Corollary 5.2.** *Let  $\varphi : \mathbb{A}_{\mathbf{Z}}^n \rightarrow \mathbb{A}_{\mathbf{Z}}^n$  be a morphism of affine spaces over  $\mathbf{Z}$ , the ring of integers. If the Jacobian  $J(\varphi)$  is equal to either  $\pm 1$ , then  $\varphi$  is an isomorphism.*

**Added in Proof.** The author would like to be grateful to Moeko Oda for walking with him along his life.

## REFERENCES

- [1] S.Abhyankar, Expansion technique in algebraic geometry, Tata Institute of Fundamental Research, Springer-Verlag (1977).
- [2] K.Adjamagbo, The complete solution to Bass generalized Jacobian Conjecture, (preprint).
- [3] K.Adjamagbo, Sur les morphismes injectifs et les isomorphismes des varieties algebriques affines, Comm. in Alg., 24(3) (1996), 1117-1123.
- [4] A.Altman and S.Kleiman, Introduction to Grothendieck duality theory, Lecture Notes in Math. 146, Springer-Verlag (1970).
- [5] M.F.Atiyah and L.G.MacDonald, Introduction to Commutative Algebra, Addison-Wesley, London (1969).
- [6] H.Bass, E.H.Connel and D.Wright, The Jacobian conjecture, Bull. A.M.S., 7(2) (1982),287-330.
- [7] L.A.Campbell, A condition for a polynomial map to be invertible, Math. Ann., 205 (1973),243-248.
- [8] E.Kunz, An Introduction to Commutative Algebra and Algebraic Geometry, Birkhäuser(1985).
- [9] R.Fossum, The Divisor Class Group of a Krull Domain, Springer-Verlag (1973).
- [10] V.S.Kulikov, Generalized and local Jacobian Problems, Russian Acad. Sci. Izuv. Math, Vol 41 (1993), No.2, 351-365.
- [11] O.H.Keller, Gamze Cremona-Transformationen, Monatsheft fur Math. und Phys., Vol. 47 (1939),299-306.
- [12] H.Matsumura, Commutative Algebra, Benjamin, New York (1970).
- [13] H.Matsumura, Commutative Ring Theory, Cambridge Univ. Press(1980).
- [14] M.Nagata, Lectures on The Fourteenth Problem of Hilbert, Tata Institute of Fundamental Research, Bombay (1965).
- [15] M.Nagata, Local Rings, Interscience (1962).
- [16] S.Oda, On finitely generated, birational, flat extensions of integral domains, Annales Math. Blaise Pascal 1 (2004),35-40.
- [17] S.Oda and K.Yoshida, A short proof of the Jacobian conjecture in the case of degree  $\leq 2$ , C.R. Math. Rep., Acad. Sci. Canada, Vol.V (1983),159-162.
- [18] S.Wang, On the Jacobian conjecture, J. Algebra 65 (1980), 453-494.
- [19] D.Wright, On the Jacobian conjecture, Illinois J. Math., 25 (1981), 423-440.

Department of Mathematics

Faculty of Education

Kochi University

2-5-1 Akebono-cho, Kochi 780-8520

JAPAN

ssmoda@kochi-u.ac.jp